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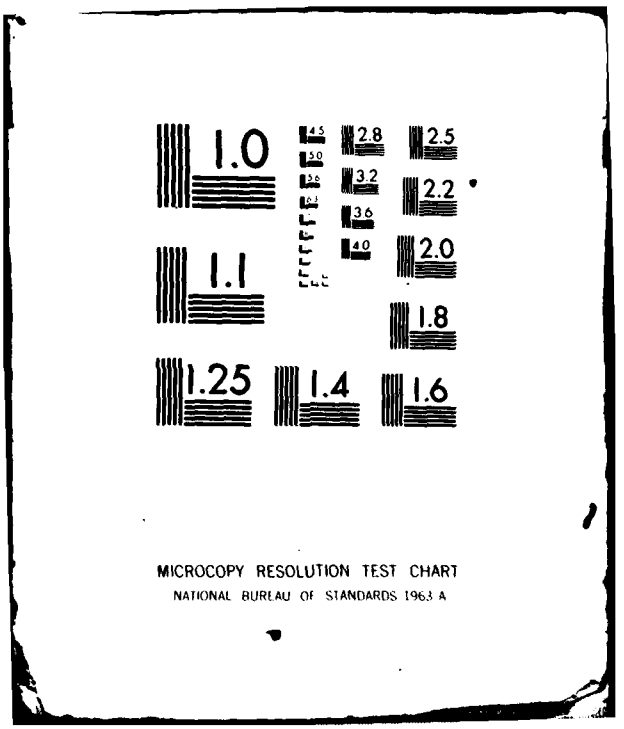
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THE DETERMINATION OF PROBABILITIES AND MOMENTS OF MEASURES OF VISIBILITY ON CURVES IN THE PLANE FOR POISSON SHADOWING PROCESSES

(14) VPI-TR-2

By

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1. Introduction

In a previous paper [10] we studied the problem of determining the distribution of the lighted portion (measure of vacancy) of the circumference of a circle having a source of light at the center and shadowing arcs generated by randomly placed disks within the circle. The centers of the disks are assumed to be uniformly distributed on the plane and their diameters are i.i.d. random variables having a general distribution independent of the location. Formulae were given for the numerical determination of all moments of the measure of vacancy. The present paper generalizes the methodology of obtaining moments of the measure of vacancy to cases where a more general type of random objects are casting shadows on star-shapped curves on the plane. We start the development in the next section by defining a class of random objects in the plane, which satisfy certain initial conditions. In particular, we consider the class of Poisson random objects for which the methodology of the present paper is developed. We introduce also in this section the basic concepts related to the shadowing process. In section 3 we derive general formulae for the probability of the simultaneous visibility of n points in the plane. These formulae are applied in section 4 to the development of recursive formula for the moments of measures of visibility of curves in the plane. Examples of the application of the methodology presented in the report, different from the one worked out in [10] will be given in following reports.

The literature on shadowing processes is quite limited. Chernoff and Daly [2] studied the distribution of length of shadows of disks on a line. The shadowing problem, however, is a special case of the general coverage problem on which there is extensive literature. In particular we refer to the studies of Robbins [5,6], Ailam [1], Siegel [7,8] and the monograph of Solomon [9] which summarizes many of the important results.

2. Random Objects and Their Shadows

Consider a countable set of disks, S, scattered on the plane. Each disk is specified by the polar coordinates (ρ,θ), $0\le\rho<\infty$, $-\pi\le\theta<\pi$, of its center with respect to an origin 0 (observation point, source of light); and by its diameter y, $0\le y<a$, $(a\le\infty)$. Let $\mathcal{R}_0$ denote the collection of Borel subsets of the sample space

$$R_0 = \{(\rho,\theta,y):\ 0\le\rho<\infty,\ -\pi\le\theta<\pi,\ 0\le y<a\}. \tag{2.1}$$

For every $B\in\mathcal{R}_0$ denote by $N\{B\}$ the number of disks such that $(\rho,\theta,y)\in B$ (satisfying condition B). If, for every $B\in\mathcal{R}_0$, $N\{B\}$ is a random variable, then the elements of S are called random disks. In particular, the disks are called Poisson random if for every $B\in\mathcal{R}_0$, $N\{B\}$ has a Poisson distribution with mean

$$\nu(B) = \mu\iiint_B G(dy|\rho,\theta)H(d\rho,d\theta) \tag{2.2}$$

where μ, $0<\mu<\infty$, is an intensity parameter; $H(\rho,\theta)$ and $G(y|\rho,\theta)$ are the sigma-finite measure of the location coordinates (ρ,θ) and the conditional c.d.f. of the diameter, y, given (ρ,θ), respectively.

The special case in which $H(d\rho,d\theta) = \rho d\rho d\theta$ and $G(y|\rho,\theta) = G(y)$, is called the standard case. The standard case is the one in which the centers of the disks are uniformly distributed on the plane and their diameters are independent of their location. In this case

$$\nu(B) = \mu\int_0^a H_B(y)dG(y), \tag{2.3}$$

where μ is the mean number of disks per unit area; $H_B(y)$ is the area of the region in which disks of diameter y, satisfying condition B, are centered.

The development of the theoretical framework is not restricted to random disks in the plane but can be generalized to Poisson random objects in the plane, or even more generally in the space, by replacing the parametric vectors (ρ,θ,y) to a general parametric vector $(\eta_1,\ldots,\eta_p)$, which characterize the location,

orientation, shape and size of a family of objects. In such cases R_0 and $\bar{R}_0$ are properly generalized. In order to simplify the presentation, we restrict attention in the present study to the family of random disks in the plane. Most of the formulae, however, are valid for general families.

A natural requirement for shadowing processes is that the source of light (the origin) is uncovered. We therefore introduce the initial condition

$$C_0 = \{(\rho,\theta,y);\ \tfrac{y}{2} \le \rho < \infty,\ -\pi \le \theta < \pi,\ 0 < y < a\} \tag{2.4}$$

and assume that all random disks satisfy C_0. In special cases one considers more stringent initial conditions, specified by sets C contained in C_0. For example, in the previous paper of Yadin and Zacks [10] the initial condition considered is

$$C = \{(\rho,\theta,y);\ \tfrac{y}{2} \le \rho < 1 - \tfrac{y}{2};\ -\pi \le \theta < \pi,\ 0 < y \le 1\} \ . \tag{2.5}$$

Let P be a point in the plane. P is said to be visible (in light) if the line segment $\overline{OP}$ does not interset any random disk: The set of all visible points in direction s, $-\pi \le s < \pi$, starting at the origin, is called a line of sight, L_s. Let P=(r,s) be a point in the plane, in direction s and distance r from the origin. A random disk (ρ,θ,y) intersects the line segment $\overline{OP}$ if, and only if, (ρ,θ,y) belongs to

$$B(r,s) = \{(\rho,\theta,y);\ 0 \le \rho \le \rho(\theta,y,r,s),\ -\pi \le \theta \le \pi,\ 0 < y < a\} \ , \tag{2.6}$$

where

$$\rho(\theta,y,r,s) = \begin{cases} r\cos(\theta-s) + \left(\left(\frac{y}{2}\right)^2 - r^2\sin^2(\theta-s)\right)^{1/2}, & \text{if } |\theta-s| \le \tan^{-1}\left(\frac{y}{2r}\right) \\ \dfrac{y}{2\sin|\theta-s|}, & \text{if } \tan^{-1}\left(\frac{y}{2r}\right) < |\theta-s| \le \frac{\pi}{2} \\ \frac{y}{2}, & \text{if } \frac{\pi}{2} < |\theta-s| \end{cases} \tag{2.7}$$

Accordingly, a point P=(r,s) is visible if $N\{B(r,s)\cap C\} = 0$. Hence, a line of sight L_s has magnitude

$$||L_s|| = \sup\{r;\ N\{B(r,s)\cap C\} = 0\} . \tag{2.8}$$

Notice that $||L_s||$ is a random variable.

Consider a curve, C, in the plane such that each ray originating at 0 intersects C at most at one point (star-shaped curve). Such a curve is specified by a continuous positive function $r(s)$, $s_L \le s \le s_U$, $-\pi \le s_L \le s_U \le \pi$, where $(r(s),s)$ are the polar coordinate of points on C. We further consider smooth curves, that is r(s) is almost everywhere differentiable. We assume that the initial condition C in C_0 implies that C is uncovered by random disks. Define the indicator function I(s), $s_L \le s \le s_U$ so that I(S)=1 if $(r(s),s)$ is a visible point and I(s)=0 otherwise. The total measure of the visible part of C is given by

$$V\{C\} = \int_{s_L}^{s_U} I(s)[r^2(s) + (r'(s))^2]^{\frac{1}{2}} ds \tag{2.9}$$

V{C} is the visibility measure of C. In coverage problems it is known also as the measure of vacancy (Ailam [1]).

3. Visibility Probabilities for Poisson Random Objects

Let P=(r,s) be a point in the plane. Under the Poisson randomness assumption, the probability that P is visible is

$$\begin{aligned} Q(r,s) &= P[N\{B(r,s)\cap C\} = 0] \\ &= \exp\{-\nu\{B(r,s)\cap C\}\} . \end{aligned} \tag{3.1}$$

From this function one can immediately obtain the distribution of $||L_s||$. Indeed,

$$P\{||L_s|| > \ell\} = Q(\ell,s), \quad 0 \le \ell < \infty. \tag{3.2}$$

Furthermore, according to (2.2) and (2.6) one obtains that

$$(3.3) \qquad \nu\{B(r,s)\cap C_0\} = \int_0^a \int_{-\pi/2}^{\pi/2} \int_{y/2}^{\rho(\theta,y,r,s)} G(dy|\rho,\theta)H(d\rho,d\theta).$$

In particular, for the standard case

$$(3.4) \qquad \nu\{B(r,s)\cap C_0\} = \mu\xi r ,$$

where ξ is the expected diameter of random disks. This implies that in the standard case the distribution of $||L_s||$ is exponential with mean $1/\rho$, where $\rho = \mu\xi$. This special result is well known and may be obtained directly from the Markovian nature of the phenomenon (see Feller [3, p. 10]).

Let $C = \{(r(s),s),\ s_L \le s \le s_U\}$ be a specified smooth curve in the plane. Let s_1 and s_2, $s_L \le s_1 < s_2 \le s_U$, be the orientation coordinates of two specified points, P_1 and P_2, on the curve. In the following development we consider only initial conditions C which specify the set of all (ρ,θ,y) points which may cast shadow on the curve C. Note that $\nu\{C\}<\infty$ for all curves of finite length. A necessary and sufficient condition for P_1 and P_2 to be simultaneously visible is that

$$N\{(B(r(s_1),s_1)\cup B(r(s_2),s_2))\cap C\} = 0 .$$

Since $B(r(s_1),s_1)$ and $B(r(s_2),s_2)$ are generally not disjoint it is more convenient to consider a union of disjoint sets obtained in the following manner. Define, for an s in $[s_L,s_U]$,

$$(3.6) \qquad B_+(s) = \{(\rho,\theta,y):\ \frac{y}{2\sin(\theta-s)} < \rho < \infty;\ s < \theta < s_U;\ 0 < y < a\}$$

and

$$(3.7) \qquad B_-(s) = \{(\rho,\theta,y):\ \frac{y}{2\sin(s-\theta)} < \rho < \infty;\ s_L < \theta < s,\ 0 < y < a\} .$$

$B_+(s)$ and $B_-(s)$ are the sets of all possible random disks on the right and on the left of a ray with orientation s, which do not intersect it. The set under consideration is the complement with respect to C of the union of the disjoint sets $B_-(s_1)\cup B_+(s_2)$, $B_+(s_1)\cap B_-(s_2)$. Accordingly, the probability that P_1 and P_2 are simultaneously visible is

(3.8)
$$P(s_1,s_2) = \exp\{-[\nu\{C\} - \nu\{(B_-(s_1)\cup B_+(s_2))\cap C\} - \nu\{(B_+(s_1)\cap B_-(s_2))\cap C\}]\} .$$

Generally, if $s_L \le s_1 < \ldots < s_n \le s_U$ are the orientation coordinates of n points on the curve C, the probability that they are simultaneously visible is

$$P(s_1,\ldots,s_n) = \exp\{-[\nu\{C\} - \nu\{(B_-(s_1)\cup B_+(s_n))\cap C\} - \sum_{i=1}^{n-1} \nu\{(B_+(s_i)\cap B_-(s_{i+1}))\cap C\}]\}.$$

4. Moments of Visibility Measures

According to (2.9) the expected values of the visibility measure, $V\{C\}$, is

(4.1)
$$E\{V\{C\}\} = \int_{s_L}^{s_U} E\{I(s)\}\ell(s)ds,$$

where $\ell(s)ds = [r^2(s) + (r'(s))^2]^{\frac{1}{2}}ds$ is the infinitesmal length of C at s and $E\{I(s)\} = Q(r(s),s)$. It may often be more convenient to compute $Q(r(s),s)$ according to (3.8), in which we substitute $s_1=s_2=s$. In this case,

(4.2)
$$Q(r(s),s) = P(s,s)$$
$$= \exp\{-[\nu\{C\} - \nu\{(B_-(s)\cup B_+(s))\cap C\}]\}.$$

Generally, for every $n\ge 2$ the n-th moment of $V\{C\}$ is given by

$$E\{V^n\{C\}\} = \int_{s_L}^{s_U} \cdots \int_{s_L}^{s_U} E\{\prod_{i=1}^{n} I(s_i)\} \prod_{i=1}^{n} \ell(s_i) ds_i \tag{4.3}$$

$$= n! \int \cdots \int_{\{s_L \le s_1 \le \cdots \le s_n \le s_U\}} P(s_1,\ldots,s_n) \prod_{i=1}^{n} \ell(s_i) ds_i$$

Indeed, $E\{\prod_{i=1}^{n} I(s_i)\}$ is the probability that all the n points are simultaneously visible. According to (3.9),

$$P(s_1,\ldots,s_n) = e^{-\nu\{C\}} \xi(s_1,s_n) \prod_{i=1}^{n-1} \psi(s_i,s_{i+1}), \tag{4.4}$$

where

$$\xi(s_1,s_n) = \exp\{\nu\{(B_-(s_1) \cup B_+(s_n)) \cap C\}\}$$

(4.5) and

$$\psi(s,t) = \exp\{\nu\{(B_+(s) \cap B_-(t)) \cap C\}\} \quad .$$

Let $Q\{C\}$ denote the probability that the whole curve C is visible. Notice that $Q\{C\} = \exp\{-\nu\{C\}\}$. From (4.3) - (4.5) we obtain

$$\begin{aligned} E\{V^n\{C\}\} = n! Q\{C\} \int_{s_L}^{s_U} ds_n \int_{s_L}^{s_U} ds_1 \cdot \ell(s_1) \xi(s_1,s_n) \int_{s_1}^{s_n} ds_2 \\ \cdot \ell(s_2) \psi(s_1,s_2) \int_{s_2}^{s_n} \cdots \int_{s_{n-3}}^{s_n} ds_{n-2} \\ \cdot \ell(s_{n-2}) \psi(s_{n-3},s_{n-2}) \int_{s_{n-2}}^{s_n} ds_{n-1} \ell(s_{n-1}) \psi(s_{n-2},s_{n-1}) \ell(s_n) \psi(s_{n-1},s_n) \end{aligned} \tag{4.6}$$

Define recursively the functions

$$\begin{aligned} \eta_1(s,t) &= \ell(s)\psi(s,t) \ , \ s_L \le s \le t \le s_U \\ \eta_k(s,t) &= \int_s^t \eta_1(s,v) \eta_{k-1}(v,t) dv. \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7) we obtain, for every $n \geq 2$,

$$E\{V^n\{C\}\} = n!Q\{C\}\int_{s_L}^{s_U}\int_{s}^{s_U}\ell(s)\xi(s,t)\eta_{n-1}(s,t)ds\,dt \tag{4.8}$$

In cases whter $B_-(s)\cap C$ and $B_+(t)\cap C$ are disjoint sets for every $s<t$ one can derive simpler expressions for the moments.

Finally, let $\ell\{C\}$ denote the length of C. The distribution of $V\{C\}/\ell\{C\}$ is concentrated on $[0,1]$, with jumps at the two end points 0 and 1 and absolutely continuous elsewhere. It follows that

$$\lim_{n\to\infty} E\{V^n\{C\}\}/\ell^n\{C\} = P\{V\{C\} = \ell\{C\}\} = Q\{C\} \tag{4.9}$$

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